Take care when using probability-weighted discount rates in capital budgeting

or: intuition's great...*except when it's not*

It's probably a fair assessment that this one belongs in the "*Not Exactly Groundbreaking But Potentially Useful*" file—assuming you even maintain such a thing. While helping a student the other day with a fairly routine capital budgeting problem, I was reminded that in situations involving **probability-weighted discount rate estimations**, one's intuitive approach might indeed lead one astray.

A rather stylized and over-simplified illustration provides the motivation, and sets the stage for a more generalized approach. Suppose for example you know you'll have an opportunity for dropping some coin into a project one year hence. This project will generate a single cash flow of 100,000[1] five years (i.e., at $t = 6$) following the initial investment. Obviously you're interested in estimating this project's present value today, perhaps to decide whether you should expend any energy pursuing further investigation.

Certainly you have no precise knowledge of what the appropriate 5-year discount rate will be, when it's time to pull the trigger on this deal (one year from today). However, you've been provided with an estimation that the best discount rate[2] will either be $r_1 = 10\%$ or $r_2 = 20\%$, each with a probability of ½. You assign to Simeon the task of pricing this project, given the available data. Just before you head out the door en route to the golf course, you recall that Simeon has exhibited questionable judgment in the past, so just for safe measure you assign this same task to Dori, to provide (hopefully) an independent confirmation of the results.

Simeon reasons that the *expected* discount rate $E(r)$ is

$$E(r) = \frac{1}{2} \cdot r_1 + \frac{1}{2} \cdot r_2 \ = 15\%$$

He then proceeds to PV the project (as of $t = 1$)…

$$100{,}000[1 + E(r)]^{-5} = 100{,}000(1.15)^{-5} \ = \ 49{,}718$$

Dori on the other hand uses a different bit of reasoning in her approach. She determines the project's PV under each of the discount-rate possibilities, and *then* applies the weights:

$$\left[\frac{100{,}000}{(1+r_1)^5}\right]\left(\frac{1}{2}\right) + \left[\frac{100{,}000}{(1+r_2)^5}\right]\left(\frac{1}{2}\right) \ = \ \mathbf{L} \ = \ 51{,}140$$

Oops. While the magnitude of the difference is barely a rounding error here, the bigger point is that the two approaches—either one of which might reasonably be seen as logical—yield different results. In a different context (greater number of cash flows? longer project life or duration? asymmetric probabilities?) the spread between the two approaches might indeed become material.

What's disconcerting is that, upon reflection, the logic used by *both* employees seems on its face to have merit, at least intuitively. Simeon prices the single cash flow using the *expected discount rate*, whereas Dori hits the set of possible values with the corresponding probability weights to derive an *expected PV*. C'mon, admit it; until you give it some thought it's not immediately clear who's got it right. Seems we'll have to put it under a bright light and examine the inner workings.

(Spoiler alert): Dori nailed it. To see this rather easily, assume that two identical projects are undertaken simultaneously. Suppose further—while suspending economic theory for a moment—that the two projects have appropriate discount rates of 10% and 20%, respectively. Keeping with the foregoing illustration, each project throws off a single cash flow of 100K, with a 5-year maturity. Then the aggregate value of the two projects together is

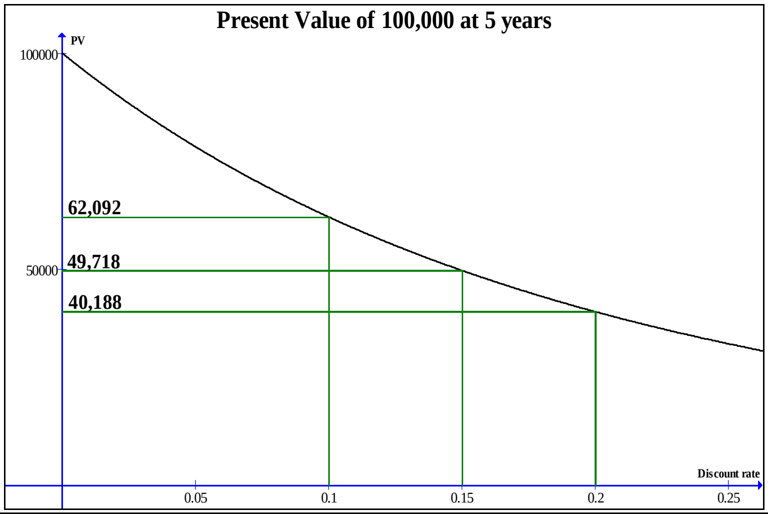
$$100{,}000(1.1)^{-5} + 100{,}000(1.2)^{-5} \ = \ 102{,}280$$

for an mean PV of **51,140** per. In fact, it's easy to see how Dori's model follows the same mathematical process, generally speaking.

So where did Simeon's approach come off the tracks? The answer rests with the implicit *linearity* of his approach, which doesn't cut it very well in the decidedly *nonlinear* world of discounting. Simply put, while 15% is indeed the simple average of 10% and 20%, the *average* value of two projects, priced with 10% and 20%, respectively, is *not* equivalent to a single project discounted at 15%. Putting it a bit differently, experiencing an actual discount rate of 10% half the time, and 20% half the time, isn't the same as experiencing an actual discount rate of 15% *all* the time, *ceteris paribus*. A simple graph showing PV of a single cash flow, as a function of discount rate, illustrates it in ***Figure 1***.

Here the symmetry between two discount rates (10% and 20%) and their simple mean is contrasted against the nonsymmetrical relationship among their corresponding PVs (courtesy of the hyperbolic PV function).

Figure 1



NERDBAR

At its core, the generic PV function is a rational function, and hence behaves as a hyperbola (more specifically, the physical constraints confine it strictly to a Quadrant I range and domain). For any constant compounding period $\boldsymbol{t}$ and any given cash flow $\boldsymbol{C_t}$, the present value of $\boldsymbol{C_t}$, as a function of the discount rate $\boldsymbol{r}$, sets up as $PV(r) = \frac{C_t}{(1+r)^t}$. The denominator is a $\boldsymbol{t}^{th}$-degree polynomial of $\boldsymbol{r}$, thus making PV a card-carrying member of the rational-function fraternity.

The unrealistic setup used as the backdrop for the discussion thus far is too simplified to have much direct real-world relevance. But I'll leave it to the reader to expand on the idea (should anyone so desire; a questionable proposition, surely) and generalize to more realistic scenarios involving multiple cash flows, asymmetry across the probability weights, and so on.

Still, the point's been made: **In a capital budgeting (or similar) application, when pricing cash flows using a range of possible discount rates—as opposed to a single assumed rate—be sure the corresponding probability weights are used to develop an expected present value from the individual state-specific PVs, rather than to develop an expected discount rate which is then used to price the cash flows.**

Having said all that, though, is there a way to salvage Simeon's method, preserving the appealing logic of his approach while also producing a correct result? Yes, indeed, if we employ the probability weights not to derive an expected discount *rate*, but rather an expected discount *factor*. In other words, let's use the discount rates in their rational form. When Simeon revises his computations in this manner (on his lunch hour, natch) his discount factor becomes…

$$(1+r_1)^{-5}\left(\frac{1}{2}\right)+(1+r_2)^{-5}\left(\frac{1}{2}\right) = \mathbf{L} = 0.5114$$

His pricing computation then results in

$$100{,}000 \times 0.5114 = 51{,}140$$

Much better. By applying the weights to the discount rates *in their rational form*—as they're used in the discounting process—rather than to their 'percentage' form, the nonlinearity is automatically taken into account. Still, Dori got it right the *first* time…

EPILOGUE

Putting Dori in direct supervision of Simeon seemed an easy choice, given her superior approach to the assigned task. Your new free time increment, from having offloaded some of your supervisory duties to Dori, was merely a side benefit. Still, no better way to exploit this unintended benefit than to run back out for another 18 holes.

While traversing the parking lot toward your waiting vehicle, a little glimmer of an idea began taking form in your mind: "Let's turn it around—does this hold true in reverse?" Tomorrow's to-do list, you remember, includes a computation of an expected **future** value of a proposed immediate investment. Again, you don't know what yield this project would actually generate, but you do have a set of possible yields along with their corresponding probability weights. Will the computation give different results, depending on whether you apply the weights to the yields, vs. to the set of possible outcomes?

Hmmm….that's one to think about—tomorrow. A quick glance at your watch says your tee time is in 20 minutes.

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[1] To reduce clutter I tend to favor omission of currency symbols, and treat them instead as simply numerical values. Feel free to insert your own $, £, ¥, or € as you like. Presto! It's an interactive document!

[2] The annualized 5-year spot rate expected to be in effect at $t = 1$.